Choice Sequences and the Continuum

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Abstract: According to L.E.J. Brouwer, there is room for non-definable real numbers within the intuitionistic ontology of mental constructions. That room is allegedly provided by freely proceeding choice sequences, i.e., sequences created by repeated free choices of elements by a creating subject in a potentially infinite process. Through an analysis of the constitution of choice sequences, this paper argues against Brouwer's claim.

Keywords: Intuitionism, Potential infinity, Choice sequences, The continuum, Platonism

There seem to be two options. *Either* you believe that mathematics is restricted to potential infinity. As the progression of natural numbers can be considered a mere potential infinity, that does not prevent you from having all the natural numbers available. Similarly, you can justify the use of rational numbers. You are also plausibly entitled to *definable* irrational numbers, as the sequence of rational numbers in a Cauchy sequence that can be defined can also be understood as merely potentially infinite, because the definition itself plays a constitutive rôle. However, you would have to banish non-definable real numbers from your stock of mathematical entities.[1]

Or you think that actual infinity is acceptable. Then you have the "full" set of real numbers at your disposal, including those that are constituted by actually infinite sequences of rational numbers. The price is that you are now committed to a much more extensive ontology.[2]

L.E.J. Brouwer thinks that with intuitionism you can have it both ways: that there is room for non-definable reals in an austere ontology of only potential infinity. Specifically, he claims that non-definable real numbers are available in the form of choice sequences that are constructed in a potentially infinite process that is random, in the sense that the creator refrains from following

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[1] This position is occupied by, among others, Bishop and Bridges (1985), Weyl (1918), and Markov (1954), with some variation on what is considered an acceptable definition. If the modern concept of real numbers had been developed in his time, Aristotle would presumably also belong here.

[2] This is the mainstream position today. If the modern concept of real numbers had been developed in his time, Plato would presumably also belong here.

a law for such construction. The purpose of this paper is to argue that he is mistaken.[3]

I will start out with a general (and uncritical) introduction to intutionism.[4] I feel that this is necessary because intuitionism is too often thought of as the *result* of applying a non-standard logic, when instead it is based on metaphysical considerations; and those metaphysical considerations (somewhat confused, as I will subsequently argue that they are) are more important to the issue of arbitrary real numbers than matters of logic are. This introduction, in section 1 below, will be redundant for the Brouwer expert, but should be useful for the reader who primarily knows Brouwer's ideas through logic-centered work on intuitionism (or not at all). Section 2 lays out the specific target claim of this paper, which is then discussed in sections 3, 4, and 5.

1 General introduction

Brouwer, the father of intuitionism, aimed to create a mathematics that avoids abstract objects and actual infinity. He did so by identifying the subject matter of mathematics with the potential infinity of certain mental constructions of a creating subject. Inspired by Kant (1787), Brouwer ontologically located mathematics in the human intuition of time.[5] According to him, the basic building block of mathematical constructions is the so-called empty two-ity, which is the result of fixing on a moment of time, noticing it giving way to another such moment, and abstracting away the contingent and specific elements of the experiences that the subject happens to have at those two. The construction of the empty two-ity gives us the numbers 1 and 2. That can be iterated by dividing the *now* of the initial two-ity's *past-now* distinction into a "new past" moment and a "new now" moment, resulting in an object *(old past-new past)-new now* that can play the rôle of the number three, and so on. According to Brouwer, the mathematical universe is limited to what can be constructed in this way.[6]

Brouwer can account for the meaning and truth of "$2 + 2 = 4$" as follows: I have constructed a two-ity, then another two-ity and then a four-ity, and succeeded in constructing a bijection between the disjoint union of the two former and the latter. That account is in terms of actual constructions. To account for the necessity of the truth of "$2 + 2 = 4$", and for the truth of "$10 \cdot 10^{100} = 10^{101}$" we have to go beyond actual constructions; but we can do that while staying within the confines of intuitionism. While I may make a mistake in an attempt to construct a truth-maker for "$2 + 2 = 4$", a mental construction can come with an *intention* to execute the construction

[3] Fictionalism (Field 1980, Balaguer 1996) is another attempt at combining a parsimonious ontology with acceptance of undefinable real numbers (and indeed everything else in classical mathematics).

[4] See van Atten (2004) and van Stigt (1990) for more thorough introductions.

[5] See page 8 and chapter 2 of Brouwer (1907).

[6] "In this way" is very vague. While some of the details will be fleshed out below, the phrase also reflects a vagueness and lack of detail in Brouwer's own papers. See Kuiper (2004) for an attempt at filling in some of the details Brouwer omitted.

in a certain way, and this intentionality implies that there is a normative aspect to constructions. This, in turn, allows us to say that any *correct* (i.e., intention-fulfilling) construction of the sum of two and two would necessarily result in four. And though I will never actually construct the mental objects that the sentence "$10 \cdot 10^{100} = 10^{101}$" is properly about *using* the intuition of time, *reflection* on the intuition of time shows the subject that the future is in principle (in some sense of "in principle") open-ended, and that the series of natural numbers could therefore in principle be extended indefinitely. It is therefore clear that even the enormous numbers referred to in this sentence are potentially constructable, and that suffices (because we can prove in advance that if they were constructed correctly, then they would relate in the way indicated by the sentence). Thus, Brouwer's mentalism provides support for a mathematics of potential infinity. But he rejects actual infinity.

According to Brouwer (1908), the mentalistic ontology also necessitates a rejection of classical logic. A simple illustration can be given using the classical proof that there exist irrational numbers a and b such that a^b is rational. It is a proof by cases: either $\sqrt{2}^{\sqrt{2}}$ is rational or irrational. If it is rational, let both a and b be equal to the irrational number $\sqrt{2}$, and then a^b is rational. If it is irrational, let a be equal to $\sqrt{2}^{\sqrt{2}}$ and let again b be equal to $\sqrt{2}$, in which case we have

$$a^b = \left(\sqrt{2}^{\sqrt{2}}\right)^{\sqrt{2}} = \sqrt{2}^{\sqrt{2}\cdot\sqrt{2}} = \sqrt{2}^2 = 2,$$

i.e., again a rational number. This proof is non-constructive, as it does not inform us which irrational number a has the sought-after property. And for Brouwer, that is an epistemic point with ontological implications: only if we have constructed an irrational number a and constructed its having the property of being equal to a rational number when raised to the power of an irrational number b, is there such a number; for there is nowhere else in all of Being to locate it than in our constructions.

The culprit in the classical proof is the very first step: the assumption that $\sqrt{2}^{\sqrt{2}}$ is either rational or irrational in the absence of a construction to support one of the disjuncts. Thus, *tertium non datur* is not in general a valid principle.

For Brouwer (1907, chapter 3; 1947; 1952), logic does not have the central position in mathematics that it has for the classical mathematician. Rather, logical laws are merely highly general descriptions of the interrelations of constructions—or, indeed, highly general descriptions of the *language* that can, imperfectly, be used to convey an essentially language-less construction from one person to another. An inference rule being valid means that, whenever constructions corresponding to its premises are at hand, a construction corresponding to its conclusion can be effected.[7]

Brouwer's non-standard ontology in general, and his revision of logic in particular, mean that a broad range of important classical theorems fail intuitionistically, including the theorem that every real number is positive or non-positive, $\forall x{\in}\mathbb{R}(x > 0 \lor x \leq 0)$. Brouwer gives examples of real numbers

[7]For more on this, see Hansen (2016).

for which we cannot assert that it is one or the other. A prerequisite for those examples is the intuitionistic notion of real numbers. With the exception of the strict finitist, all parties to the debate agree that a real number is an infinitary object. Either it is an ordered pair of actually infinite sets of rational numbers (Dedekind 1872), or an actually infinite, converging sequence of rational numbers (Cauchy 1821; Heine 1872)—or, if you ask Brouwer, a potentially infinite, converging sequence of rational numbers.[8] That is, a real number is the process whereby a creating subject constructs more and more elements of a so-called choice sequence. The elements can be freely chosen by the subject, or he can decide to follow a rule when choosing elements. In the latter case, it must be possible to calculate each element in a finite amount of time for which an upper bound is known in advance.

The specific details of the definition of "real number" can be filled out in several different, intuitionistically acceptable ways. For the purpose of this paper, let us define a real number as a choice sequence $\langle q_1, q_2, q_3, \ldots \rangle$ of rational numbers, such that $|q_n - q_{n+1}| \leq 2^{-n}$ for all natural numbers n, and such that each q_n is of the form $m \cdot 2^{-n-1}$ for some integer m.

A real number that, at present, can neither be asserted to be positive nor to be non-positive can be constructed using a so-called fleeing property, defined by Brouwer (1955, 114) as follows.

> A property f having a sense for natural numbers is called a *fleeing property* if it satisfies the following three requirements:
>
> (i) For each natural number n, it can be decided whether or not n possesses the property f;
>
> (ii) no way is known to calculate a natural number possessing f;
>
> (iii) the assumption that at least one natural number possesses f, is not known to be contradictory.

An example of a fleeing property P is, for a given finite sequence of digits not yet found in the decimal expansion of π and not yet proved not to occur in it, that that sequence occurs beginning at the n'th decimal. Then, let the real number a be defined as the choice sequence that begins with the elements $1/4, -1/8, 1/16, \ldots, (-1/2)^{n+1}, \ldots$, and continues like that as long as no n has had the property P, and stays constant at $(-1/2)^{n+1}$ from the first n that has the property P onwards (if such an n is found). Then, at any given point in the construction where the choice sequence is still "oscillating", the creating subject is not in possession of a truth maker for either the sentence "$a > 0$" or the sentence "$a \leq 0$".

This invalidity of a classical theorem leads to the validity of a non-classical one: namely, that all functions from $\mathbb{R}$ to $\mathbb{R}$ are continuous (Brouwer 1924).

[8] In the second and third cases it should really be "an actually/potentially infinite equivalence class of actually/potentially infinite, converging sequences of rational numbers". In the interest of avoiding cumbersome formulations, I will pretend that a sequence *is* a real number, rather than an element of one.

For instance, this is an illegitimate definition of such a function,

$$f(x) = \begin{cases} 0 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0, \end{cases}$$

because it would have to map a to a choice sequence $f(a)$. The first two elements of $f(a)$ could both be equal to $\frac{1}{2}$, for that is consistent with subsequent elements of $f(a)$ converging to 0, and also consistent with subsequent elements of $f(a)$ converging to 1. However, as we cannot make it the case that $a > 0$ or $a \leq 0$ via a finite calculation with an upper bound on time consumption that is known in advance, there is no way, at present, to choose a third element of $f(a)$, for any possible choice would either be too far away from 0 or too far away from 1 to allow the sequence to converge to that value if a subsequently attains a specific value (i.e., if a natural number is determined to have the property P, or it is determined that it is impossible for any natural number to have it). Thus f is not a *total* function on the real numbers, but only a partial function defined for those real numbers that are either positive or non-positive.

According to Brouwer, the space of real numbers is not limited to those that are governed by a rule, as a is. By allowing for sequences that are not, the creating subject is supposed to be capable of producing *additional* real numbers: non-definable real numbers that can saturate the continuum.

2 Brouwer on freely proceeding choice sequences

The aspect of Brouwer's intuitionism that distinguishes it the most from other types of constructivism is its use of choice sequences: sequences created in time via successive choices of new elements by a creating subject (Brouwer 1952, 142). At any point in time, only a finite initial segment has been constructed. The sequence is, therefore, never finished, but always in a state of expansion. According to Brouwer, basing mathematics on such objects eliminates the need to assume that something actually infinite exists.

The creating subject can choose to pick the elements according to an algorithm: for example, one that selects rational numbers that are increasingly better approximations of π. But that is not required. The subject can also create a sequence in which each choice of an element is made at random. The subject may grant himself the freedom to allow each element to be *any* member of some species (i.e., class), for instance, the species of natural numbers—or may elect, from the beginning of the construction or at any point during it, to impose restrictions on his own future choices. An important example is the decision to create a real number. This amounts to the subject self-imposing the restriction that each element shall be a rational number q_n of the form $m \cdot 2^{-n-1}$, satisfying $|q_{n-1} - q_n| \leq 2^{-(n-1)}$ if $n > 1$.

Some terminology: a choice sequence governed by an algorithm will be called "lawlike", whereas one that is not will be called "freely proceeding". Although this seems like a simple distinction, some further clarification is required. First, the creating subject may impose some restrictions on future choices, but without going so far that there is only one option for each element. Sequences characterized by such restrictions will still be called "freely proceeding" even though that freedom is partial. Second, a choice sequence may be governed by, for example, a law that each element is the sum of the corresponding elements of two freely proceeding sequences. Since this does not qualify as an algorithm, when everything is taken into account, such a sequence will be considered freely proceeding. However, complicated examples like these are not very relevant to my purposes, so the reader is encouraged to keep a simpler stereotype of arbitrary choice in mind when freely proceeding sequences are discussed. Third, the categorization of choice sequences into lawlike and freely proceeding is intended to be exclusive and exhaustive. That exhaustiveness is achieved when one category is defined from the other by negation is not obvious when the negation is intuitionistic. But the categories are meant to be time-relative; i.e., a sequence is considered freely proceeding at a given instant if the subject has not, at that instant, decided to follow an algorithm for the rest of the sequence. Thus, at every instant, each sequence is either lawlike or freely proceeding, although a freely proceeding sequence may become lawlike later.

According to the platonist, there are among the abstract mathematical objects real numbers that cannot be defined. While disagreeing with classical mathematics in many other respects, including whether abstract objects exist, Brouwer also claimed to have found a place for undefinable real numbers in the intuitionist ontology, namely, among the freely proceeding choice sequences. This thesis is perhaps presented most clearly in the following passage:

> [Intuitionism] also allows infinite sequences of pre-constructed elements which proceed in total or partial freedom. After the abandonment of logic one needed this to create all the real numbers which make up the one-dimensional continuum. If only the predeterminate sequences of classical mathematics were available, one could by introspective construction only generate subspecies of an ever-unfinished countable species of real numbers which is doomed always to have the measure zero. To introduce a species of real numbers which can represent the continuum and therefore must have positive measure, classical mathematics had to resort to some logical process, starting from anything-but-evident axioms[. . .]. Of course, this so-called complete system of real numbers has thereby not yet been created; in fact only a logical system was created, not a mathematical one. On these grounds we may say that classical analysis, however suitable for technology and science, has less mathematical reality than intuitionist analysis, which succeeds in structuring the positively-measured continuum from real numbers by admitting the species of freely-proceeding convergent infinite sequences of rational numbers and without the need to resort to language or logic. (Brouwer 1951, 451–452)[9]

[9]See also Brouwer (1930) and (1952).

So, Brouwer's claim is that the free creation of sequences—an arbitrary choice of an element, followed by another arbitrary choice of an element, *ad infinitum in potentia*—can result in sequences that cannot be defined. Without relying on abstract objects, but just the human potential for free mental construction, the intuitionist has access to the "full" set of real numbers. It is this notion that I want to dispute.

There are two slightly different ways to interpret it. The stronger interpretation is that Brouwer does, in one crucial respect, exactly the same thing as the classical mathematician, by finding a non-denumerable totality of points with which to *identify* the continuum. If so, Brouwer changed his mind in his late work, because in his early years, before he came up with the idea of freely proceeding choice sequences, he was of the opinions that the continuum is a primitive notion; that it cannot be constructed out of entities of any other type; and that, specifically, it cannot be identified with a set of points. His 1907 description of the continuous and the discrete holds that they are complementary and equally basic aspects of the Primordial Intuition, and that points (and numbers) can only be used to analyze a pre-existing continuum by being the endpoints of the subintervals into which it could be decomposed.[10] One reason Brouwer gives for why a continuum cannot be a set of points is that the available points are only those that can be identified with rational numbers or definable real numbers, i.e., lawlike sequences, implying that there is only a denumerable infinity of them and hence not enough to exhaust the continuum (Brouwer 1913). However, it is also possible to interpret the above passage in a weaker way: instead of Brouwer's claim being that his reals make up the one-dimensional *intuitive* continuum, they just make up the *mathematical* continuum, i.e., the best possible model we can have of the intuitive continuum. That would be consistent with this model falling short of perfection. In this interpretation, Brouwer makes a more modest claim, namely that the freely proceeding sequences add to the model something that the lawlike sequences cannot. However, the subtle differences between these exegetical theses do not affect the critique made below.

3 Constitution of free choice sequences

Several claims discussed up to this point have been prefaced with "according to Brouwer", and for most of those that were not, it was implicit. That ends now, as I will move into a critical mode to seek a more precise answer to the question of what a freely proceeding choice sequence is, independently of Brouwer's position. I will reach an answer to that question of constitution at the end of this section, and then, in section 4, argue on the basis of that answer that Brouwer is mistaken in thinking that freely proceeding choice sequences can contribute something to the analysis of the continuum that lawlike sequences cannot.

The project of determining the constitution of freely proceeding choice sequences independently of Brouwer's position is a little dicey, since, as the

[10] This is, of course, a view originating with Aristotle (1930).

inventor of the term, Brouwer has some authority over the meaning of "freely proceeding choice sequence". So let me clarify the rules of the game. I do not believe that it would be reasonable to say that, if there are no entities with all the properties Brouwer claims for freely proceeding sequences, then there are no freely proceeding sequences. Instead, I think that a basic characterization can serve as a common ground by picking out a certain class of entities, whose more sophisticated properties we can then disagree about. Let the first two paragraphs of the previous section serve as that basic characterization.

So what exactly constitutes a freely proceeding choice sequence? As is witnessed by the debate on personal identity, questions of constitution can often be elucidated by first asking related questions of individuation and self-identity over time. So, if I begin a freely proceeding sequence of natural numbers now at t_1 by making the first element 4, and then *now* at t_2 add 9 to it as its second element, what is it that makes the sequence at t_1 identical to the sequence at t_2?

The strongest possible answer, that they are qualitatively identical, can quickly be ruled out. If they were qualitatively identical they would have exactly the same properties, and so would already at t_1 have 9 as its second element. So, by the same token, for each n, at t_1 it would be a property of the sequence that there was some specific number that was its nth element. But then, the sequence would be actually infinite.

Instead of the relevant property being *has 9 for its second element*, it could be *has, at* t_2 *and later, 9 for its second element*. But this makes little difference; the problem still arises, *mutatis mutandis*, because there are still an actual infinity of properties. The fact that some of them are about the future does not make a meaningful difference. Brouwer cannot accept that what will happen in the future corresponds, in general, to facts in the present—at least not when one assumes the possibility of an infinite future with genuinely random events; yet, Brouwer needs that premise if his choice sequences are to play the rôle of non-definable real numbers. Hence, he is committed to anti-realism with respect to the future.

The failure of the foregoing attempt to reach a satisfactory answer teaches us two things: that we must look for some criterion of numerical identity instead, and that such a criterion must allow for the sequence to be genuinely dynamic. This is acknowledged by Brouwer (1955, 114):

> In intuitionist mathematics a mathematical entity is not necessarily predeterminate, and may, in its state of free growth, at some time acquire a property which it did not possess before.

However, commenting on this quote, van Atten (2007, 14) states that

> a property such as 'The number n occurs in the choice sequence x' is constitutive of the identity of x, but is generally undecidable and does not satisfy PEM [the principle of the excluded middle].

If van Atten's statement were true, the property *the number 9 occurs in the choice sequence* α would be constitutive of α, but that would imply that the t_1-incarnation of α is not α. Consequently, diachronic self-identity of a choice sequence would be impossible. At most, the property *the number* n *occurs in the choice sequence* x being constitutive of the identity of x is the case only from the point in time at which n is added to the sequence. On pain of commitment to actual infinity, it cannot be before that. And from that time onwards, it is decided.[11]

To avoid actual infinity in both its explicit and implicit forms, do we need to conclude that the temporal instantiation of our freely proceeding sequence at t_2 is the object

$$\langle 4, 9 \rangle?$$

No, for that is just an ordered tuple, and a choice sequence is not just *that*. There is a dynamic aspect to a sequence that is absent from the n-tuple. This difference is, however, not in the past; the tuple has also been created, one element added at a time, in a temporal process. In Brouwer's universe there are no atemporal mathematical objects; it is just that some of the temporal objects have been completed. That is the difference between the tuple and the sequence: the former has found its final form, while the latter will continue to undergo changes.

This is, however, exactly the kind of claim that we must be cautious about interpreting. The notion that it "will continue to undergo changes" must not be understood as an assertion about the actual future of the sequence, for the actual future does not exist. Given the commitment to anti-realism with regard to the future, the only content this claim can have is that the creating subject has an *intention* to amend the sequence. So, allowing "intention to expand" to be short for "intention to expand in keeping with the restriction ..." if there is a restriction, the following is a more promising proposal regarding the constitution of our freely proceeding sequence at t_2:

$$\langle 4, 9, \text{intention to expand} \rangle$$

Under that proposal, the present product of an ongoing construction is merely what has actually been constructed plus the psychological fact that its creator does not consider it finished. The self-identity of the sequence being created over time does not rely on any objects in the future, but simply on the subject choosing, when he adds a new element, to consider the extended finite sequence a part of the same freely proceeding sequence as the old one.[12]

I think this is the correct answer, that is, it is the closest thing we can find in "the inventory of the world" to what Brouwer envisions a freely proceeding choice sequence to be. Nevertheless, in the next section it will be

[11] Van Atten has informed me that he only intended to say that if the third element of α has been chosen to be 1, then it is known that a choice sequence β, for which something different from 1 has been chosen as its third element, is not equal to α.

[12] One might deny that such a decision to identify really has the force to secure actual identity. But then, we would launch into an even more extensive disagreement with Brouwer, so I will not argue against it here.

useful to contrast this answer with another possible answer, namely that the constitution of our choice sequence at t_2 looks like this:

$$\langle 4, 9, x_3^{\mathbb{N}}, x_4^{\mathbb{N}}, x_5^{\mathbb{N}}, \ldots \rangle$$

Here, $x_n^{\mathbb{N}}$ is supposed to be an *indeterminate element* that is restricted to $\mathbb{N}$. That is, at t_2 it is true that the third element (e.g.) is a natural number, but neither true nor false that it is equal to 7. Then, at t_3, the choice sequence may change to

$$\langle 4, 9, 7, x_4^{\mathbb{N}}, x_5^{\mathbb{N}}, x_6^{\mathbb{N}}, \ldots \rangle,$$

as the next choice determines the third element, which until then was indeterminate.

I think that $\langle 4, 9, \text{intention to expand} \rangle$ is the correct answer to the question of the constitution of the choice sequence at t_2 because it captures all of what seem to be the facts of the situation under a parsimonious ontological analysis thereof. That is, 4 has been chosen as the first element, 9 has been chosen as the second element, and the creating subject has an intention to continue expanding the sequence—that's it! It is a simple situation and there is no need to invoke the existence of mysterious indeterminate objects to understand it.[13] Hence, let us refer to it as the "simple answer" and to the alternative answer as the "indeterminacy answer". However, I will consider both of these answers to the question of the constitution of freely proceeding choice sequences in the following section.

4 Evaluation of Brouwer's claim

Let us evaluate Brouwer's claim that he has succeeded in supplying an adequate ontology for the "full" system of real numbers that includes non-definable sequences of rational numbers, in light of the above analysis of the constitution of a freely proceeding choice sequence.

A preliminary point is that, at any given time, only a finite number of choice sequences actually exist, because a choice sequence only exists if someone has created it. Thus, relying only on choice sequences that actually exist will definitely not suffice; rather $\mathbb{R}$ must consist of all *possible* choice sequences that satisfy the definition given above. And that is the idea: where the classical, platonic reals, by virtue of the hierarchical nature of the set-theoretical universe, must all exist for the set of them to do the same, Brouwer only

[13] I am overstating the simplicity a bit. If you and I each produce a choice sequence and we have so far, by chance, picked the same elements in the same order, and we both intend to expand our respective sequences according to the same restrictions, if any, we have nevertheless produced different sequences. (The two sequences will be *equal* (so far), but not *identical*. Brouwer also makes this distinction, for example in his definition of "species" (1952, 142). Troelstra (1977) makes the distinction using the terminology "extensional identity" and "intensional identity".) That is not captured by "$\langle 4, 9, \text{intention to expand} \rangle$"; there are also concrete facts about who the creator of the sequence is, when it was started, etc., that belong in a complete analysis of the constitution of a choice sequence. However, this complication is irrelevant to the issue at hand, for there is still no need to invoke the existence of indeterminate objects.

commits himself to the possibility of constructing each of his reals. They do not all have to exist prior to them being collected in the species of all reals. His continuum is the totality of all possible convergent sequences of rationals. I will not take issue with that. Instead, the question to be asked is: which choice sequences are really possible?

Assume that a is a platonic real number, i.e., that a is an actually infinite (and converging) sequence of rational numbers $\langle a_1, a_2, \ldots\rangle$; and assume further that this sequence is undefinable. If a creating subject attempts to construct the same real number ("same" in a mathematical, but not an ontological sense), it is possible for him to construct $\langle a_1, \text{intention to expand}\rangle$, after which it is possible to expand it to $\langle a_1, a_2, \text{intention to expand}\rangle$, and then to $\langle a_1, a_2, a_3, \text{intention to expand}\rangle$. However, at any given instant, only a finite initial segment of a has been created.

Assuming for the moment that they exist, the actually infinite, undefinable sequences do not correspond to possible routes for potentially infinite choice sequences: "possible" means "can be taken", and the *entire* route corresponding to a platonic undefinable sequence can never be taken, only initial segments of it.

There is a nice metaphor of Posy's (1976, 98–99) that we can make use of here. He likens choice sequences to the route of a bus traveling on a forking highway. The journey of the bus can be seen from different perspectives. First, there is the perspective of a passenger seated with his back to the bus driver, so that he can only see the route already traversed. Second, there is the perspective of the driver, who in addition to the knowledge possessed by his passenger has an intention regarding where to travel to from his present position. And third, there is the perspective of a helicopter pilot looking down on the bus and road system from above, seeing both the traveled path and the roads ahead. Given the rejection of actual infinity, there is no helicopter perspective. Actually infinite roads are no less actually infinite than completed infinite travels. The only legitimate perspectives are the passenger's and the driver's: the former being finitely extensional and the latter both finitely extensional and finitely intensional. For the bus driver or the creating subject, there is an infinity of possibilities in the indefinite future. But one must not conflate an infinity of possibilities with the possibility of infinity.

If that was a bit too metaphorical, the same point can also be made more formally, either by a comparison with classical mathematics or by employing tense logic. Assuming classical mathematics, we can say that freely proceeding choice sequences can only deliver the elements of $\mathbb{N}^{<\omega}$ (or $\mathbb{Q}^{<\omega}$, etc.), not the elements of $\mathbb{N}^{\omega}$ (or $\mathbb{Q}^{\omega}$, etc.). Or with the notation of tense logic, we can disambiguate what has been conflated by intuitionism. An intuitionist would say that when a creating subject is constructing a freely proceeding choice sequence a of rational numbers, then $\forall n{\in}\mathbb{N}\,\exists q{\in}\mathbb{Q}\,(a_n = q)$ is true. But it is not, in any straightforward way, true of the present. Taking a cue from Prior (1967), we can see that there must be an implicit "it will be the case that"-operator somewhere.[14] Writing this operator as "F", we

[14]This operator has previously been employed by Niekus (2010) to help clarify aspects

can disambiguate $\forall n{\in}\mathbb{N}\,\exists q{\in}\mathbb{Q}\,(a_n = q)$ as either $\forall n{\in}\mathbb{N}\,F(\exists q{\in}\mathbb{Q}\,(a_n = q))$ or $F(\forall n{\in}\mathbb{N}\,\exists q{\in}\mathbb{Q}\,(a_n = q))$.[15] Only the former is true, but it is the truth of the latter that would be required for a to be (or to become) a real number, as opposed to an always expanding yet always finite sequence.[16]

When platonism and actual infinity have been rejected, there is no *sub specie aeternitatis* perspective under which the process of extending one finite sequence to another finite sequence again and again can constitute an omega-sequence. If only finitely many terms have been added to the sequence at any given time, if future choices are not predetermined by a law, and if the future does not exist, there is simply no sense in which the sequence has the infinitely many terms that would allow it to be an irrational number.

My conclusion is that choice sequences cannot do the same work that the classical set of real numbers allegedly do. Does that conclusion change if we replace the simple answer with the indeterminacy answer? I think that, underlying Brouwer's claim, there is a vague intuition that it does: each individual indeterminate element "ranges", in some "fuzzy" way, over all the natural numbers (or over the members of some other species)—"it is not true that $x_4^{\mathbb{Q}} = 7/8$, but it is also not false"—and thus, the infinitely many indeterminate elements collectively range over all sequences of natural numbers in a way that is not restricted by what can be defined.

However, I don't see how that intuition can be substantiated. It is vague, in that Brouwer's official account of mathematical ontology offers no support for the assumption of there being indeterminate objects. And he is (what would otherwise count as) quite explicit in his delimitation of the mathematical realm: only mental constructs are admitted, and indeed, only those that can be introduced in accordance with one of the two "acts of intuitionism" (Brouwer 1952). The first act of intuitionism is the purification of mathematics, whereby everything that cannot be grounded in the intuition of time is exorcised. The intuition of time gives the subject the awareness of a difference in the form of the before-after relation, or in Brouwer's words, the so-called Primordial Intuition of the empty two-ity. As explained in section 1, this can be translated into the numbers 1 and 2, and the number 3 can be created by holding onto a before-after relation while distinguishing it collectively from a new "after". By repetition, the natural numbers can be constructed, as can any finite object or set of finite objects equipped with relations and operations, in a manner that is not very different from the classical approach. The second act of intuitionism is the realization by the creating subject that he is not limited to already-created mathematical objects, but free to employ the Primordial Intuition in any way he likes, in a temporarily unbounded "free unfolding of the empty two-ity". This clears the space for choice sequences: the subject can set out to make a potentially infinite sequence consisting of "mathematical entities previously acquired".

of intuitionism.

[15]Please note that F here only disambiguates between potential and actual infinities of identity facts, and not between $\mathbb{N}$ being potentially and actually infinite.

[16]The branch of formal intuitionism in which I think it should have been most obvious that this conflation happens is in Beth's semantics (Beth 1964, 444ff.), which is precisely an attempt at capturing the semantics of sequences of choices.

The second act is what makes Brouwer's universe potentially infinite instead of finite. Importantly, however, it is a potential infinity *of Primordial Intuition-created entities*. That is, the second act does not sanction a new kind of basic object that is indeterminate, but only allows for the open-ended addition and combining of more and more determinate mental constructs.

Even if we set this aside, the points made in the case of the simple answer still stand. First, in the attempt to construct a, the creating subject can construct $\langle a_1, x_2^{\mathbb{Q}}, x_3^{\mathbb{Q}}, x_4^{\mathbb{Q}}, \ldots\rangle$, and then determine the value of $x_2^{\mathbb{Q}}$ to get $\langle a_1, a_2, x_3^{\mathbb{Q}}, x_4^{\mathbb{Q}}, x_5^{\mathbb{Q}}, \ldots\rangle$, and then determine the value of $x_3^{\mathbb{Q}}$ to get $\langle a_1, a_2, a_3, x_4^{\mathbb{Q}}, x_5^{\mathbb{Q}}, x_6^{\mathbb{Q}}, \ldots\rangle$, etc.; but he never produces something that is equivalent to a itself. Second, the tail of indeterminate elements must be considered a potential infinity, and that precludes the infinitely many elements from being independent of each other in the way needed for them to collectively have a range that includes a non-definable sequence. There is no equivalent of the arbitrary platonic real number a, even if we pretend to believe that there are such things as indeterminate elements.

Let me consider a possible objection. An intuitionist might bite the bullet and accept that no freely proceeding choice sequence is the equivalent of a, but claim that this is because such sequences are so fundamentally different from the objects of classical mathematics that there is no direct correspondence—and then proceed to claim that the freely proceeding choice sequences nevertheless "fill up the holes" in the continuum that remain after only the lawlike sequences have been poured into it.

While this defense seems contrary to the spirit of the Brouwer quote in section 2, the objector may try to draw some support from Troelstra (1977, section 2.5), according to whom a freely proceeding choice sequence must be extensionally different (see footnote 13) from any sequence that is intensionally different. While Troelstra's claim only ranges over intuitionistic sequences, it doesn't seem too much of a stretch to say that if a freely proceeding choice sequence cannot be extensionally identical to any other intuitionistic sequence, then it also cannot be extensionally identical to a platonic sequence (or, at least, it doesn't seem too much of a stretch if one, for the sake of argument, is sufficiently eclectically minded to allow for such comparisons between intuitionistic and platonic objects).

This defense can be overcome by reformulating my critique of freely proceeding sequences. Instead of comparing it to classical sequences, we can instead point out that a freely proceeding choice sequence cannot add anything to the constitution of the continuum that cannot already be accomplished by rational numbers and lawlike choice sequences. Notice that the definition of "real number" given earlier implies that a real number is a convergent sequence of rational numbers in which each element restricts all subsequent elements to an increasingly smaller interval around it, and each such interval must also be included in the previous intervals. So, in a freely proceeding choice sequence that is meant to be a real number (i.e., the creating subject restricts himself to choices that are in conformity with the definition), when n elements have been chosen, the first $n-1$ elements no longer carry any relevant information. This is because the nth element indicates which interval future choices are restricted to, and all the earlier intervals include the nth interval and therefore do not restrict the creating subject any further. As such, it makes no difference to the theory of real numbers if we identify the development

t_1: $\langle 1, \text{intention to expand} \rangle$,
t_2: $\langle 1, 1/2, \text{intention to expand} \rangle$,
t_3: $\langle 1, 1/2, 3/4, \text{intention to expand} \rangle$

with

t_1: $\langle 1, \text{intention to change} \rangle$,
t_2: $\langle 1/2, \text{intention to change} \rangle$,
t_3: $\langle 3/4, \text{intention to change} \rangle$,

where "intention to change" may likewise be short for something of the form "intention to change in keeping with the restriction ...". At any given time, the mathematical content of a freely proceeding sequence equals an interval with rational endpoints. The creating subject is just changing his mind about which interval to use, and each choice is one that could have been made initially, if it were not for the restriction that each choice must be of the form $m \cdot 2^{-n-1}$. The implication of this is that, contrary to Brouwer's claims,[17] freely proceeding choice sequences do nothing that rational numbers cannot do.

The intuitionist may respond to this claim with another objection, namely by pointing out that the species of freely proceeding choice sequences of rational numbers is uncountable, while the species of rational numbers is not. Thus, the former species has a crucial property in common with the platonic set of real numbers, suggesting that it does contribute something to the continuum that the latter species does not. However, a closer look at the reason for the uncountability reveals that conclusion to be unfounded.

The proof goes as follows. Let f be a function from the species of freely proceeding sequences of rational numbers to $\mathbb{N}$. According to the intutionistic notion of function, it must be possible, in the case of each freely proceeding sequence, to approximate the function value to any desired degree of precision based on knowledge of some finite number of that sequence's elements.[18] Therefore, since there is a minimum distance between different natural numbers (namely 1!), it must be possible to determine the function value *exactly* from knowledge of some finite number of elements of the sequence. Let α be a freely proceeding choice sequence, and n be the number of initial elements thereof that are needed to determine $f(\alpha)$. There is a possible freely proceeding choice sequence, β, that is extensionally different from α but shares the first n elements. Since the function value can be determined solely on the

[17] See Brouwer (1930) for an overview of what the freely proceeding choice sequences are supposed to contribute.

[18] This demand on functions is closely related to what is demanded of real numbers; see the last few paragraphs of section 1.

basis of those n elements, $f(\beta) = f(\alpha)$. Ergo, there is no injective function from the species of freely proceeding sequences of rational numbers to $\mathbb{N}$, implying that the former is uncountable.

One point in favor of this objection is that it does not turn on any contentious properties of freely proceeding sequences. That is, the proof also goes through under my thesis about the constitution of such sequences: β does not have to be actually infinite; it only needs the n elements that are shared with α, plus one more to make it differ from α.

The problem with the objection lies elsewhere. To articulate it, we need to first analyze why the uncountablity of the choice sequences seems, prima facie, to indicate that they contribute something to the modeling of the continuum. Let us say that there are two species, S_1 and S_2, of mathematical entities that represent points on the continuum. Assuming that S_2 does not contain any "duplicates" in the form of entities that represent the same point, I would have to concede that S_2 contributes something to a mathematical model of the continuum that S_1 does not, if there are *more* entities in S_2 than in S_1. And that is what the proof seems to show for the two species it concerns.

However, I do not think that the intuitionistic proof of the uncountability of the choice sequences demonstrates that there are more choice sequences than there are rational numbers *in the relevant sense of "more"*. Let me explain why. What has been proved above is that the species of freely proceeding choice sequences has a higher cardinality than the species of rational numbers. "Higher cardinality" is a precise technical term, defined via existence and lack of bijections. I use "more" in a pre-theoretic sense—it is an intuitive concept that could potentially be explicated in a satisfying way by a precisely defined technical term. The relevant question here is whether "higher cardinality" is a good explication of "more". I would submit that it only is so under one condition, and that that condition is satisfied in a classical, platonic universe but not in the intuitionistic one.[19] In a platonic universe, whenever s_1 and s_2 are two sets of the same size (understood pre-theoretically), then, by the "magic" of combinatorial set creation that requires no human acts, one element from s_2 can be made the function value of an element from s_1; another element from s_2 can be made the function value of another element from s_1; etc., "until" all the elements from both are used up. Assuming the existence of platonic sets, it should be possible to use up both sets without repetition if and only if s_1 and s_2 are the same size. That is, under the condition of the existence of the platonic universe of sets, sameness of size guarantees the existence of a bijection, but only because no restrictions are placed on how the function can be put together. Under those circumstances it would seem that "more" can be understood in terms of higher cardinality. If, on the other hand, you reject some combinatorial functions, you may reject the *witnesses* to sameness of size, but that does not rule out sameness of size. By demanding that it is always possible to calculate function values to an

[19]Actually, it is not essential to the argument that the condition would be satisfied in a platonic universe; it just makes for a nice contrast for the purpose of explaining the situation in the intuitionistic universe.

arbitrary degree of approximation in finitely many steps, intuitionism places severe restrictions on the function concept, an thus, intuitionistic uncountability loses any direct connection with the intuitive concept of size. That a species is intuitionistically uncountable means that intuitionism does not recognize any bijection between that species and $\mathbb{N}$, but does not necessarily mean much more than that.[20]

Let me drive the point home with a somewhat silly comparison. Let us say that J.E.L. Qrouwer has invented a new foundation for mathematics, qintuitionism. He accepts the classical rational numbers, and in addition, he has a collection of numbers he calls "q-numbers". To the rest of the world, q-numbers do not seem to be essentially different from rational numbers, but Qrouwer insists that they are. He claims that the collection of q-numbers is uncountable and that they can therefore model the continuum much better than the rational numbers. But when the claim is investigated by a non-qintuitionist, she discovers that this is just because Qrouwer has a very narrow function concept. It only allows q-numbers to be mapped to natural numbers in such a way that the function value is equal to or smaller than the denominator in the irreducible fraction that (according to everyone but Qrouwer) the q-number is equal to. Hence, there is no qintutionistically acceptable bijection from the collection of q-numbers to $\mathbb{N}$; but that does not imply anything about the size of the collection. In other words, the uncountability is due to the qintuitionistic impoverishment of mathematics with respect to the function concept, and not to an enrichment of mathematics with respect to modeling of the continuum.

Obviously, there are huge differences between freely proceeding choice sequences and the q-numbers of this little fairytale, but it does show that one cannot infer directly from the uncountability-according-to-some-restrictive-notion-of-function of some collection of mathematical entities to a claim about that collection being essentially richer than the collection of rational numbers. Thus, it would have to be explained how freely proceeding choice sequences are *relevantly* different from q-numbers. I would argue that they are not, because—again—a freely proceeding choice sequence that is supposed to be a real number in effect amounts to no more than the creating subject repeatedly changing his mind about which rational number to use.[21] At any rate, that would hold true unless and until the subject turned it into a lawlike sequence, in which case it might cease to be a rational number; but then it just becomes a *definable* irrational number.

[20] Note that none of this is intended to suggest that classical mathematics is the (or *a*) correct mathematics. It just means that if classical mathematics is not, then the intuitive notion of size can probably not be captured by the notion of cardinality. While one *may* take that as a reason to prefer classical mathematics, one can also go in the opposite direction and adopt the anti-Cantorian position that there is no (non-trivial) notion of infinite size to capture. As a third option, one may consider the notion to be primitive.

[21] Hence, the assumption behind my concession above is also false: the two freely-proceeding choice sequences $\langle 0, 1/2, \text{intention to expand}\rangle$ and $\langle 1, 1/2, \text{intention to expand}\rangle$ are "duplicates" in the sense that they represent (at the given point in time) the same point on the continuum (and there is no atemporal fact about what point they represent).

5 Final considerations

Our conclusion so far is that we cannot account for the existence of freely proceeding choice sequences that have all the properties claimed by Brouwer based on entities that we have independent reasons to believe exist.[22] The final option available to a metaphysisian who finds himself in that kind of situation is to simply postulate the existence of the kind of entity that he wants as basic components of the world, irreducible to anything else. That is not an uncommon move among philosophers of mathematics who support classical mathematics: they will often take sets to be fundamental entities that are not constituted by anything physical or mental (or at most, partially constituted by something physical or mental in the case of impure sets containing such things). Could a Brouwerian do something similar?

If that move is made, then it is no longer clear what the point of intuitionism is. It is supposed to deliver a mathematics that is free of actual infinity and, because based on immediate intuitions, epistemically transparent. But if a freely proceeding choice sequence can constitute an undefinable real number, then it is actually infinite; and being deemed a fundamental object does not change that. And if it is a fundamental object that cannot be analyzed completely in terms of mental acts that we are familiar with, then we do not have the direct epistemic access to it that Brouwer claims.

If one is willing to make that kind of move, why not just make it in the direction of classical, platonic mathematics? The price is the same (acceptance of alien objects to which direct epistemic access is lacking); and the reward is a stronger and more well-behaved mathematics satisfying *tertium non datur*.

But if one is not willing to make that move, I conclude that one has to look for a kind of mathematics that does without undefinable real numbers. Bishop's brand of constructivistic mathematics (Bishop and Bridges 1985) may be suitable for someone who wants to retain Brouwer's mentalism, Brouwer's rejection of actual infinity, and Brouwer's verificationism, while accepting this critique of Brouwer's intuitionism. As mentioned earlier, other options include Weyl's (1918) predicativism and the Russian school of constructivism founded by Markov (1954).

[22] Van Atten (2007) has reached the opposite conclusion. I find his line of reasoning to be extremely obscure. If one reads pages 89–93 in isolation, it would seem that van Atten reaches essentially the same conclusion as I did in section 3 (although couched in the more flowery language of the phenomenological tradition). But he nevertheless agrees with Brouwer, based on a vague and unsubstantiated claim that the "inexhaustibility and non-discreteness of the (intuitive) continuum" somehow fits together with the undecidability of the extensional identity of freely proceeding choice sequences of intervals (p. 87). Apparently, the idea is that the use of nested intervals delivers the non-discreteness, but the classical real numbers can also be defined in terms of sequences of nested intervals, and they are discrete (i.e., the continuum is identified with a set of points). The difference to classical mathematics is supposed to be that the sequence is unfinished. But if it is unfinished, it has a last term (at any given time) and that last term represents an interval of positive length with rational endpoints. Further, it seems like the undecidability is supposed to match the inexhaustibility. That is, the (alleged) poverty of facts about whether pairs of sequences are identical or different is expected to deliver richness of ontological structure. I do not understand how.

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